THE USE OF A MAXIMUM RATE OF DISSIPATION CRITERION TO MODEL BEAMS WITH INTERNAL DISSIPATION

A Thesis

by

MIN SEOK KO

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2004

Major Subject: Mechanical Engineering

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Approved as to style and content by:

Arun Srinivasa (Chair of Committee)	Kumbakonam Rajagopal (Member)
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ABSTRACT

The Use of a Maximum Rate of Dissipation Criterion to Model

Beams with Internal Dissipation.

(May 2004)

Min Seok Ko, B.S., Inha University

Chair of Advisory Committee: Dr. Arun Srinivasa

This thesis deals with a systematic procedure for the derivation of exact expression for the frequency equation of composite beams undergoing forced vibration with damping. The governing differential equations of motion of the composite beam are derived analytically for bending and shear deformation. The basic equations of Timoshenko beam theory and assumption of maximum rate of dissipation are employed. The principle involved is that of vibration energy dissipation due to damping as a result of deformation of materials in sandwich beam. The boundary conditions for displacements and forces for the cantilever beam are imposed and the frequency equation is obtained. The expressions for the amplitude of displacements are also derived in explicit analytical form. Numerical results of the displacement amplitude in cantilever sandwich beam varying with damping coefficient are evaluated.

To my father in heaven:

Without whom I could have never come so far

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TABLE OF CONTENTS

LIST OF TABLES

TABLE Page

LIST OF FIGURES

FIGURE Page

CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

1.1 Beam theories

In this thesis, we consider a straight bar that undergoes three fundamental types of motions: extension, shear, and bending. Each type of deformation is described by a displacement field consisting of a single independent variable defined as the bar's axial displacement of cross-sections in the case of extension, rotation of cross-sections for torsion, and transverse displacement in the case of flexure. A bar that carries loads by undergoing flexural deformation is commonly referred to as a beam.

There are a number of beam theories that are used to represent the kinematics of deformation. To describe beam theories, we introduce the following coordinate system. The x, y, and z coordinates are taken along the length, the width, and the thickness of the beam, respectively. All applied loads and geometry are such that the displacement (u, v, and w) along the coordinates (x, y, and z) are functions of x and time, t. (See Fig. 1) In the current development we assume that the kinematical quantities do not vary in the y direction.

1.1.1 Classical beam theory

The most commonly used beam theory is the Bernoulli-Euler classical beam theory.

Thesis style and format follow that of Computer Methods in Applied Mechanics and Engineering.

We shall refer to this description as classical beam theory. The fundamental kinematical assumption for classical beam theory is that planar cross-sections maintain their shape and remain perpendicular to the centroidal axis as the beam undergoes deformation.

Hence, the displacement can be expressed as

$$u = u_0 - z\frac{\partial w}{\partial x}, \quad w = w_0, \tag{1.1}$$

where u_0 and w_0 are the axial and transverse normal components of the displacement of points on neutral axis of the beam

As a result of this assumption, the displacement of any point in the beam is kinematically related to the displacement of centroid. It is clear that for the displacement field in Eq. (1.1) all strains except ε_{xx} are zero. This theory of bending is widely used for thin beams.

1.1.2 Timoshenko beam theory

The beam theory used in this thesis is the Timoshenko beam theory [18], which is based on that plane sections originally perpendicular to the longitudinal axis of the beam remain plane, but not necessarily perpendicular, to the longitudinal axis of the beam. The displacement can be expressed as

$$u = u_0 - z\theta, \quad w = w_0, \tag{1.2}$$

where ϑ denotes total rotation of a cross section as shown Fig. 1.

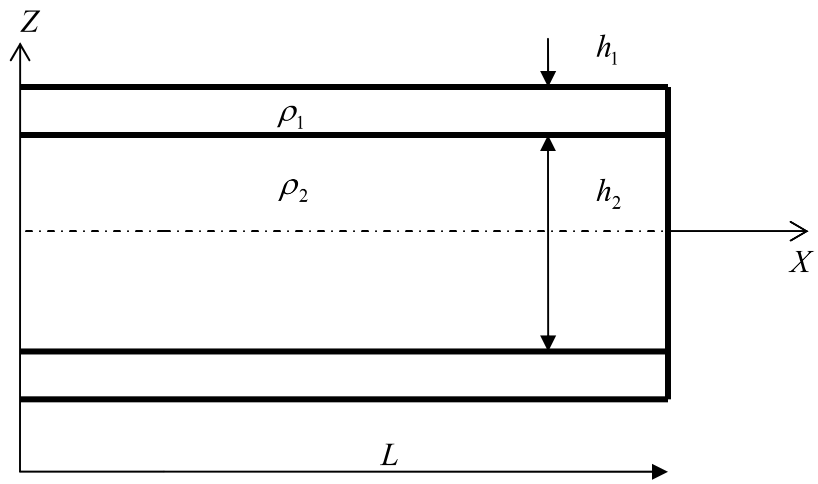

Fig. 1. Geometry and material parameters of sandwich beam

One notion that is contained in classical beam theory is that although cross sections carry a resultant shear force, the associated shear stress distribution is not accompanied by a shear strain. In actuality, the shear stress and strain vary over the cross-section, because the shear stress must be zero at the upper and lower surfaces of the bar [1]. Timoshenko approximated the effect of shear as an average over the cross-section. This involves allowing each cross-section to rotate independently of the slope of the centroidal axis in the deformed state. As part of the correction to classical beam theory, rotatory inertia of cross-sections is also incorporated into the formulation.

1.2 Sandwich structures

The number of applications for sandwich structures is steadily increasing. The term sandwich structure here refers to a structure consisting of two thin faces bonded to a thick and light weight core. The faces are typically aluminum or some composite laminate. The core could be lightweight foam or a honeycomb structure.

The bending of sandwich beams and plates is often described by means of some simplified model. The papers by Hoff [2] are one of the first fundamental works on the bending and buckling of sandwich plates. In that paper, Hamilton's principle is used to derive the differential equations and boundary conditions for the bending and buckling of sandwich plates subjected to transverse loads and edgewise compression. The buckling load is calculated for a simply supported plate subjected to edgewise compression.

Nilsson [9] developed a more general description of the bending of sandwich beams. In this model, the transverse and longitudinal wave propagations are used to describe the displacement in the core. Boundary conditions are not discussed. Renji *et al.* [13] derived a governing differential equation with consideration of shear effects of sandwich panels. Maheri and Adams [5] used the Timoshenko beam equations to describe flexural vibrations of sandwich structures. In particular, effects of variations of the shear coefficient are discussed for obtaining satisfactory results.

Various finite element methods are often proposed for describing the vibration of sandwich panels. Shi and Lam [16] presented a finite element vibration analysis of composite beams based on Hamilton's principle. Liew *et al.* [4] used a finite element model for the numerical evaluation of the frequency response functions of honeycomb panels.

There are a large number of methods describing the vibration of sandwich beams. However, the influence of energy dissipation and the expressions for mode shapes in explicit form are not sufficiently discussed.

1.3 Energy dissipation

Experience shows that the amplitude of a vibrating beam diminishes with time and that the vibrations are gradually damped out. However, the amplitude of free vibration in a purely elastic beam remains constant with time. In the case of forced vibrations, the amplitude can grow without limit at resonance in the theory. However, we know that there is always some finite amplitude of steady-state response, even at resonance, because of damping. Among all sources of energy dissipation, the case in which the damping force is proportional to velocity gradient, called viscous damping, is the simplest to deal with mathematically. For this reason resisting forces of a complicated nature are replaced by equivalent viscous damping. This equivalent damping is determined in such a manner as to produce the same dissipation of energy per cycle as that produced by the actual resisting force [20].

Study of the dynamic behavior of structures requires knowledge of their damping characteristics. The damping factor plays a major role in the assessment of structures and materials for their performance under vibration and noise control circumstances. It is common practice to apply surface treatments in the form of viscoelastic layers to structures in order to improve their damping characteristics. One such treatment is the well-known constrained layer damping [8]. One approach to analyze the constrained layer damping in beams is by using sandwich beam theory.

The use of sandwich beams with viscoelastic damping material has become common for vibration control. The three layer damped beam arrangement consists of a layer of viscoelastic material with an additional layer of elastic material bonded to the outer surface of the viscoelastic material, thus creating a three-layer sandwichtype structure with a viscoelastic damping core. There are two primary methods for dissipating energy in the viscoelastic core of a constrained layer damping treatment-shear deformation and compressional deformation [17]. Shear deformation results when the constraining layer and outer layers move parallel to each other, acting to shear the viscoelastic core. Compressional or extensional deformation results when the constraining layer and the base structure move perpendicular to each other, acting to compress or stretch the viscoelastic material.

The theory of damped structures where mechanical energy is dissipated by shearing has been thoroughly investigated. Kerwin [3] focused on mathematical modeling of long, simply supported beams with soft viscoelastic cores and thin, stiff constraining layers. He established equations which describe the viscoelastic behavior under steady vibrations of a complex shear modulus assigned to the damping layer. Ungar [19] derived for the loss factor of axially uniform linear composite structures in terms of properties of the constituents. Ross et al. [14] closely examined the definition of loss factor in terms of energy, particularly for highly damped composite structures. Mead and Markus [6] derived the sixth–order differential equation of motion in terms of the transverse displacement for a three-layer sandwich beam with a viscoelastic core. This equation was first derived, and various boundary conditions were also expressed in terms of transverse displacement. Yan and Dowell [21] deduced a simply linear equation governing the vibrations of sandwich beams. The damping characteristics of a constrained layer sandwich can also be studied by using complex elastic constants in the frequency domain. All these developments have been focused on steady sinusoidal oscillations. However, these methods can not apply to the problems which contain non-sinusoidal oscillations. In order to do this, we need explicit characterization of dissipative forces.

In order to do this, we follow the maximum rate of dissipation assumption presented by Rajagopal and Srinivasa [11, 12] to model the energy dissipation of the system by specifying the way the energy is stored and dissipated. The viscoelastic response is determined by a stored energy function that characterizes the elastic response from the 'natural configuration' and a rate of dissipation function that describes the rate of dissipation due to the viscous effects. Rajagopal and Srinivasa [11, p. 971] stated the rate of dissipation as:

> The isothermal form of the energy balance equation then stipulates that the rate of dissipation is equal to the difference between the ratio of work and the rate of change of the Helmholtz potential.

Ziegler [22] introduced the assumption of the maximum rate of dissipation. He stated that the rate of dissipation is the same as the product of the absolute temperature and the entropy production in an isothermal process. Rajagopal and Srinivasa [12] developed the assumption of maximum rate of dissipation and presented the consequences of the assumption. Rajagopal and Srinivasa [12, p. 207] describe this with the following statement:

> The way in which the current natural configuration result in dissipative behavior that is determined by a 'maximum rate of dissipation' criterion subjected to the constraint that the difference between the stress power and the rate of change of the stored energy is equal to the rate of dissipation. By choosing different forms for the stored energy function and the rate of dissipation function, a whole plethora of energetically consistent rate type models can be developed.

We will use the concept of the maximum rate of dissipation to derive governing equations of motion.

1.4 Objective

The objective of the present study is to obtain and investigate the bending response behavior of sandwich beam with energy dissipation. Although there are numerous studies on sandwich beams, few researchers have studied the steady-state solutions of sandwich beam analytically, including energy dissipation. The differential equations of motion of the composite beam will be derived analytically using the assumption of maximum rate of dissipation. Exact closed-form solutions and the numerical results will be presented. This knowledge of analytical solution and numerical results of the sandwich beams having energy dissipation will help researchers and designer to gain an insight into the damping effect that significantly affects the amplitude of sandwich beam vibration.

CHAPTER II

GOVERNING EQUATIONS OF MOTION

2.1 Modeling

2.1.1 Sandwich beam

Generally, a honeycomb sandwich beam consists of hexagonal honeycomb cores and two thin plates as shown Figs. 2 and 3. For simplicity we treat this sandwich beam as a continuum body when considering flexural vibration globally [15]. The core rows in one direction are different from those in the perpendicular direction. Therefore, we assume that the in-plane properties must be different in the each perpendicular direction. The out-of-plane properties may be different from the in-plane properties. Hence we regard the honeycomb cores as an orthotropic material.

Fig. 2. Feature of honeycomb structure

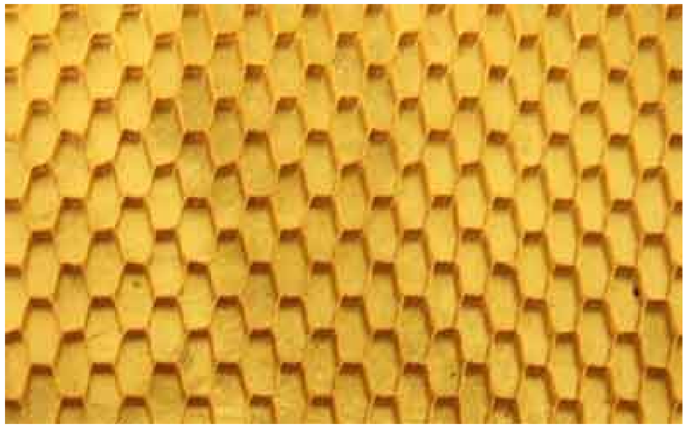

Fig. 3. Type of honeycomb core

2.1.2 Mathematical modeling

Fig. 4 depicts Timoshenko's kinematical approximation of the displacement field based on Eq. (1.2). (See Chapter 1.1.2).

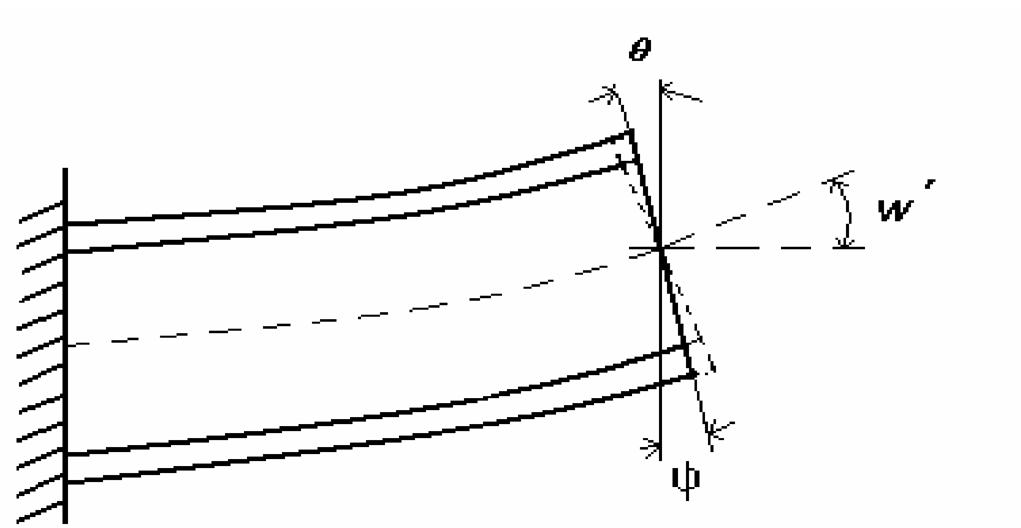

Fig. 4. Kinematics of deformation according to Timoshenko beam theory

We assume that the shear deformation in the upper and bottom layer is negligible because the thickness of both layers is much smaller than that of the core. The angle ψ measures the rotation of a cross-section relative to the normal to the centroidal axis. It is referred to as the shear angle because ψ is zero in classical beam theory. The angle θ is the total rotation of a cross-section. As in the classical beam theory, w is the transverse displacement of the centroidal axis, so $\partial w / \partial x$ is the rotation the cross-section would undergo if there was no shear deformation. In the derivation that follows, the dot and prime indicates a partial derivative with respect to the time variable t and position variable x, respectively. It follows from Figure 1 that

$$\psi = w' - \theta . \tag{2.1}$$

Based on Timosheko's approximation, the axial displacement of a point at distance z from the centroidal axis is

$$\mathrm{u} = -\,\theta(x,t)\,z\ , \tag{2.2}$$

The corresponding axial strain is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -\theta' z\ . \tag{2.3}$$

In small deformation, the shear strain is one-half the angle by which an infinitesimal rectangular element of material distorts to a parallelogram.

$$\varepsilon_{xz} = \frac{1}{2}\psi = \frac{1}{2}(w' - \theta)\,. \tag{2.4}$$

2.2 Derivation of the governing equations of motion

In this chapter we will derive partial differential equations of a sandwich beam. The motion is characterized by a displacement field whose value is a function of position as well as time.

2.2.1 Kinetic energy and potential energy

The basic quantities required to form the governing equations are the system's kinetic energy and potential energy.

The velocity is obtained by differentiating Eq. (1.2)

$$v = \dot{x}\underline{i} + \dot{z}\underline{k} = (\dot{\theta}\, z)\,\underline{i} + \dot{w}\,\underline{k}\ , \tag{2.5}$$

where $\underline{i}$ and $\underline{k}$ are unit vectors in x and z direction, respectively.

The kinetic energy of the beam is defined by

$$T\ = \frac{t}{2}\int_V \rho_i v \cdot v dV\ , \tag{2.6}$$

where t and ρ are thickness and density of beam, respectively.

From Eqs. (2.5) and (2.6), the kinetic energy in middle layer is given by

$$T_1 = \frac{t\rho_2}{2}\int_0^L \int_{-a}^{a} (\dot{\theta}^2 z^2 + \dot{w}^2)\, dZdX = \frac{t\rho_2}{2}\int_0^L (\frac{2a^3}{3}\dot{\theta}^2 + 2a\dot{w}^2)\, dX\ , \tag{2.7}$$

where $a = \dfrac{h_2}{2}$.

Similarly, the kinetic energy in upper and bottom layers is given by

$$T_2 = \frac{t\rho_1}{2}\int_0^L \int_{-b}^{-a} (\dot{\theta}^2 z^2 + \dot{w}^2)\, dZdX + \frac{t\rho_1}{2}\int_0^L \int_a^b (\dot{\theta}^2 z^2 + \dot{w}^2)\, dZdX$$

$$= \frac{t\rho_1}{2}\int_0^L 2[(\frac{b^3 - a^3}{3})\dot{\theta}^2 + (b-a)\dot{w}^2]\, dX\,, \tag{2.8}$$

where $b = a + h_1$.

Hence, the total kinetic energy of the beam is

$$T = T_1 + T_2$$

$$= \frac{t}{2}\int_0^L (\alpha_1\dot{\theta}^2 + \alpha_2\dot{w}^2)\, dX\,, \tag{2.9}$$

where

$$\alpha_1 = \rho_2(\frac{2a^3}{3}) + 2\rho_1(\frac{b^3 - a^3}{3})\,, \tag{2.10}$$

$$\alpha_2 = \rho_2(2a) + 2\rho_1(b-a)\,. \tag{2.11}$$

In the case of a purely linear elastic material, The strain energy is defined by

$$U = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\, dV\,. \tag{2.12}$$

We can express the stress-strain relation in an orthotropic core layer as follows.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}, \tag{2.13}$$

where C_{ij} are elastic constants of honeycomb cores.

We can also express the stress-strain relation in an isotropic upper and bottom layers as follows.

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{bmatrix}, \qquad (2.14)$$

where λ and μ are the Lamé constants [7].

The Lamé constants are related to the shear modulus G, Young's modulus E, and Poisson's ratio ν by

$$\mu = G = \frac{E}{2(1+\nu)}, \ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \qquad (2.15)$$

We can give the stress-strain relation in an orthotropic core layer as follows.

$$\sigma_{xx} = C_{11}\varepsilon_{xx}, \ \sigma_{zz} = C_{21}\varepsilon_{xx}, \ \tau_{zx} = 2C_{66}\varepsilon_{xz}. \qquad (2.16)$$

The stress-strain relation in an isotropic upper and bottom layers is

$$\sigma_{xx} = (\lambda+2\mu)\varepsilon_{xx}, \ \sigma_{xx} = (\lambda+2\mu)\varepsilon_{xx}, \ \tau_{zx} = 2\mu\varepsilon_{xz}. \qquad (2.17)$$

Substituting Eqs. (2.16) and (2.17) into Eq. (2.12), the strain energy in middle layer is derived as

$$U_1 = \frac{t}{2}\int_0^L \int_{-a}^{a} (\sigma_{xx}\varepsilon_{xx} + 2\tau_{zx}\varepsilon_{zx})\, dZdX = \frac{t}{2}\int_0^L \int_{-a}^{a} [C_{11}\theta'^2 z^2 + C_{66}(-\theta + w')^2]\, dZdX$$

$$= \frac{t}{2}\int_0^L [C_{11}(\frac{2a^3}{3})\theta'^2 + C_{66}(2a)(-\theta + w')^2]\, dX\,. \qquad (2.18)$$

Similarly, the strain energy in upper and bottom layers is derived as

$$U_2 = \frac{t}{2}\int_0^L \{\int_{-b}^{-a}[(\lambda + 2\mu)\theta'^2 z^2 + \mu(-\theta + w')^2]dZ$$

$$+\int_a^b [(\lambda + 2\mu)\theta'^2 z^2 + \mu(-\theta + w')^2]dZ\}dX$$

$$= \frac{t}{2}\int_0^L [2(\lambda + 2\mu)(\frac{b^3 - a^3}{3})\theta'^2 + 2\mu(b - a)(-\theta + w')^2]dX\,. \tag{2.19}$$

The total strain energy of the beam is expressed as follows.

$$U = U_1 + U_2$$

$$= \frac{t}{2}\int_0^L [\alpha_3\theta'^2 + \alpha_4(\theta - w')^2]dX\,, \tag{2.20}$$

where

$$\alpha_3 = C_{11}(\frac{2a^3}{3}) + 2(\lambda + 2\mu)(\frac{b^3 - a^3}{3})\,, \tag{2.21}$$

$$\alpha_4 = C_{66}(2a) + 2\mu(b - a)\,. \tag{2.22}$$

2.2.2 Energy dissipation

In a thermomechanical process, work done on the system is either as stored as the stored energy or dissipated. Rajagopal and Srinivasa [11] defined a rate of dissipation function as the function representing the rate of conversion of the mechanical power supplied into heat.

Let E denote the stored energy and ξ the rate of dissipation associated with the material. In an isothermal process, the ratio of dissipation ξ is equal to the difference between the ratio of work and the rate of change of the stored energy.

$$\xi = \dot{W} - \dot{E}\,, \tag{2.23}$$

where $\dot{W}$ is the mechanical power.

We define the mechanical power over the entire cross-section as

$$\dot{W} = \int_A \sigma_{ij}\dot{\varepsilon}_{ij}\ dA = \int_A (\sigma_{xx}\dot{\theta}\, z - \tau_{xz}(\dot{\theta} + \dot{w}'))\ dA$$

$$= M \cdot \dot{R} + Q \cdot \dot{S}\,. \tag{2.24}$$

with

$$M = \int_A z\sigma_{xx} dA, \quad Q = -\int_A \tau_{xz} dA\,, \tag{2.25}$$

where $R = \theta'$ and $S = -(\theta + w')$ are the curvature and shear strain of the beam, M and Q are moment and shear force, and *dA* is cross-sectional area of the beam, respectively.

It follows from Eqs. (2.23) and (2.24) that

$$\xi = Q\dot{S} + M\dot{R} - \dot{E}\,. \tag{2.26}$$

We shall make constitutive assumption for the stored energy E from Eq. (2.20), and rate of dissipation ξ. We assume that

$$\xi = f\ (\dot{R}, \dot{S})\,. \tag{2.27}$$

Now, we shall invoke the assumption of maximum rate of dissipation [12]. To carry out this maximization, we introduce the auxiliary function, Ψ using the technique of Lagrange multiplier subjected to the constraint, Eq. (2.27)

$$\Psi = Q\dot{S} + M\dot{R} - \dot{E} + \lambda f(\dot{R}, \dot{S})\,. \tag{2.28}$$

By using standard methods of the calculus of constrained maximization, we extremize ξ with respect to $\dot{S}$, $\dot{R}$, and λ respectively and obtain

$$\frac{\partial \Psi}{\partial \dot{S}} = Q - \frac{\partial E}{\partial S} - \lambda \frac{\partial f}{\partial \dot{S}} = 0\,, \tag{2.29}$$

$$\frac{\partial \Psi}{\partial \dot{R}} = M - \frac{\partial E}{\partial R} - \lambda \frac{\partial f}{\partial \dot{R}} = 0\,, \tag{2.30}$$

$$\frac{\partial \Psi}{\partial \lambda} = f(\dot{R}, \dot{S}) = 0\,, \tag{2.31}$$

Then, taking inner product of Eqs (2.29) and (2.30) with $\dot{S}$ and $\dot{R}$ respectively and using Eqs. (2.26), λ is obtained as

$$\lambda = \frac{\xi}{(\frac{\partial f}{\partial \dot{S}} \dot{S} + \frac{\partial f}{\partial \dot{R}} \dot{R})}\,. \tag{2.32}$$

Since we are interested in dissipation due to curvature, we assume that the rate of dissipation function is of the form

$$\xi = \eta \dot{R} \cdot \dot{R}\,, \tag{2.33}$$

where η is material constant representing viscosity then substitution of Eq. (2.33) into Eq. (2.32) gives $\lambda = 1$. Also substitution of the value of λ and Eq. (2.33) into Eqs. (2.29) and (2.30) reveals that

$$Q = \frac{\partial E}{\partial S}\,,\ \ M = \frac{\partial E}{\partial R} + \eta \dot{R}\,. \tag{2.34}$$

Substituting Eq. (2.34) into (2.29), we can find the generalized force-generalized displacement relations.

The results are

$$Q = \alpha_4(-\theta + w') \tag{2.35}$$

and

$$M = \alpha_3 \theta' + \eta \dot{\theta}' . \tag{2.36}$$

2.2.3 Governing equations

The equations of motion can be derived from the equilibrium equations. In component form we have

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} = \rho \ddot{u} , \tag{2.37}$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} = \rho \ddot{w} , \tag{2.38}$$

Let us start with Eq. (2.38). Integrating over the cross-section area of the beam and inserting $\sigma_{zy} = \sigma_{xy} = 0$, we have

$$\frac{\partial}{\partial x} \int \sigma_{zx} dA + \int \sigma_{zz} dy = \rho \int \ddot{w} dA . \tag{2.39}$$

Integrating Eq. (2.39) and using Eq. (2.25), we get

$$\frac{\partial Q}{\partial x} - f = \rho A \ddot{w} , \tag{2.40}$$

where *f* represents a normal force that is uniformly distributed along the y-direction.

Eq. (2.38) is multiplied by z and integrated over the beam cross-section area.

$$\frac{\partial}{\partial x} \int z \sigma_{xx} dA + \int z \frac{\partial \sigma_{xz}}{\partial z} dA = \int \rho \ddot{\theta} z^2 dA . \tag{2.41}$$

We define the moment of inertia as

$$I = \int z^2 dA . \tag{2.42}$$

The second term on the left-hand side of Eq. (2.41) shows that

$$\int \left(\frac{\partial}{\partial z} (z \sigma_{xz}) - \sigma_{xz} \right) dA = \int (z \sigma_{xz} |_b - z \sigma_{xz} |_{-b}) dy + Q . \tag{2.43}$$

Since the shear stress σ_{xz} is zero on the face $z = b$ and $z = -b$, we get

$$\int (z \frac{\partial \sigma_{xz}}{\partial z}) dA = Q \, . \tag{2.44}$$

Substituting Eqs. (2.42) and (2.44) into Eq. (2.41), we have

$$\frac{\partial M}{\partial x} + Q = \rho I \ddot{\theta} \, , \tag{2.45}$$

Using the force-displacement relations in Eqs. (2.35) and (2.36) to eliminate *M* and *F*, we now obtain two coupled partial differential equations of motion.

$$\alpha_1 \ddot{\theta} - \alpha_3 \theta'' + \alpha_4 \theta - \alpha_4 w' - \eta \dot{\theta}'' = 0 \, , \tag{2.46}$$

$$\alpha_2 \ddot{w} + \alpha_4 \theta' - \alpha_4 w'' = f \, . \tag{2.47}$$

The differential equations and force-displacement relations appearing above are the basis for analysis of flexural vibrations.

Considering the shear deformation in the upper and bottom layers and assuming the rate of dissipation function as $\xi = \eta \dot{S} \cdot \dot{S}$, we obtain the other coupled partial differential equations of motion

$$\beta_1 \ddot{\theta} - \beta_3 \theta'' - \beta_4 w'' + \beta_5 \theta - \beta_5 w' + \eta \dot{\theta} - \eta \dot{w}' = 0 \, , \tag{2.48}$$

$$\beta_2 \ddot{w} + \beta_4 \vartheta''' + \beta_5 \theta' + \beta_6 w^{iv} - \beta_5 w'' + \eta \dot{\theta}' - \eta \dot{w}'' = 0 \, , \tag{2.49}$$

where

$$\beta_1 = \rho_2 (\frac{2a^3}{3}) + 2\rho_1 (a^2 b - a^3) \, , \tag{2.50}$$

$$\beta_2 = \rho_2 (2a) + 2\rho_1 (b - a) \, , \tag{2.51}$$

$$\beta_3 = C_{11}(\frac{2a^3}{3}) + 2(\lambda + 2\mu)(a^2 b - a^3)\,, \tag{2.52}$$

$$\beta_4 = (\lambda + 2\mu)(a^3 + ab^2 - 2a^2 b)\,, \tag{2.53}$$

$$\beta_5 = 2aC_{66}\,, \tag{2.54}$$

$$\beta_6 = 2(\lambda + 2\mu)(\frac{b^3 - a^3}{3} - ab^2 + a^2 b)\,, \tag{2.55}$$

If a harmonic variation of w and θ, with circular frequency ω, is assumed, then

$$w(x,t) = W(x)\exp(i\omega t)\,,\ \theta(x,t) = \Theta(x)\exp(i\omega t)\,, \tag{2.56}$$

where *W(x)* and $\Theta(x)$ are the amplitude of the harmonically varying transverse displacement and bending rotation.

Substituting Eq. (2.56) and into Eqs. (2.48) and (2.49) gives

$$C_1\Theta(x) + C_2 W'(x) + C_3\Theta''(x) - C_3 W'''(x) = 0\,, \tag{2.57}$$

$$C_4\Theta'(x) - C_3\Theta'''(x) + C_5 W(x) + C_2 W''(x) - C_3 W^{iv}(x) = 0\,, \tag{2.58}$$

where

$$\begin{array}{l} C_1 = -\beta_1\lambda^2 + \beta_5 + i\omega\eta,\ C_2 = -(\beta_5 + i\omega\eta),\ C_3 = -\beta_3, \\ C_4 = -\beta_4,\ C_5 = -\beta_2\omega^2,\ C_6 = -\beta_5 + i\omega\eta, C_7 = -\beta_6 \end{array}, \tag{2.59}$$

Eqs. (2.57) and (2.58) can be combined into one equation by eliminating $\Theta(x)$ to give a sixth order equation in W(x) as follows.

$$W^{(vi)} + A_1 W^{(iv)} + A_2 W'' + A_3 W = 0\,. \tag{2.60}$$

where

$$A_1 = \frac{C_1C_7 - 2C_2C_4 + C_3C_6}{C_3C_7 - C_4^2},\quad A_2 = \frac{C_1C_6 - C_2^2 + C_3C_5}{C_3C_7 - C_4^2},\quad A_3 = -\frac{C_1C_5}{C_3C_7 - C_4{}^2}\,. \tag{2.61}$$

This equation has the sixth order form with different constants A_1, A_2, and A_3 to that in reference [22] which defined the shear and bending stiffness including loss factor as $G = G'(1+i\eta)$ and $D = D'(1+i\eta)$. The constants expressed in Nilsson [10] are

$$\bar{A}_1 = \frac{Gh_2D_1 - 2D_2I_p\omega^2}{-2D_1D_2}, \quad \bar{A}_2 = \frac{D_1m + 2D_2m + h_2I_pG}{-2D_1D_2}\omega^2, \quad \bar{A}_3 = \frac{h_2G\omega^2 + I_p\omega^4}{-2D_1D_2}, \quad (2.62)$$

where

$$D_1 = (1+i\eta)(\frac{E_1h_2}{12} + E_2(\frac{h_1h_2^2}{2} + h_2h_1^2 + \frac{2h_1^2}{3})), \quad D_2 = (1+i\eta)(\frac{E_1h_2^3}{12}), \quad (2.63)$$

$$I_p = \frac{\rho_2h_2^3}{12} + \rho_1(\frac{h_1h_2^2}{2} + h_2h_1^2 + \frac{2h_1^2}{3}), \quad m = 2h_1\rho_1 + h_2\rho_2, \quad (2.64)$$

in which E_1 is Young's modulus in upper layer, m is mass per unit area, and I_p is the mass moment of inertia per unit width. These comparative constants values are presented in Table 1 in result section.

2.3.4 Boundary conditions

Solution of field equations (2.46) and (2.47) requires specification of the boundary conditions. These are drawn from the geometric conditions and from natural conditions that the internal forces resultants must satisfy. Natural boundary conditions apply whenever the internal forces are known a priori. In any system either the value of a generalized coordinate may be specified, or the value of the associated generalized force may be specified, but not both. Hence, at each ends, the boundary conditions are that either the shear force or displacement is specified by

$$kGA(w' - \theta) \text{ or } w \tag{2.65}$$

and that either the rotation or bending moment is defined by

$$\theta \text{ or } EI\theta'. \tag{2.66}$$

For cantilever beam type boundary conditions, fixed-free end conditions, the four equations of constraint require that at the clamped end,

$$w|_{x=o} = 0, \ w'|_{x=0} = 0, \tag{2.67}$$

and at the free end

$$\theta'|_{x=L} = 0, \ (w' - \theta)|_{x=L} = 0. \tag{2.68}$$

CHAPTER III

EIGENSOLUTION

3.1 Case I: Uniform load

Let the beam in Fig. 4 be subjected to a general harmonic excitation carrying a uniform load of intensity F_1 along the horizontal, given in complex form as

$$f = F_1 e^{i\lambda t}, \tag{3.1}$$

where λ is the angular frequency.

Considering only steady-state forced vibrations, we assume solutions in complex form to be

$$w(x,t) = W(x)\exp(i\lambda t),\ \theta(x,t) = \Theta(x)\exp(i\lambda t), \tag{3.2}$$

where $W(x)$ and $\Theta(x)$ are the amplitude of the harmonically varying transverse displacement and bending rotation.

Substituting Eq. (3.2) into Eqs. (2.46) and (2.47) gives

$$(-\alpha_1\lambda^2 + \alpha_4)\Theta(x) - (\alpha_3 + i\lambda\eta)\Theta''(x) - \alpha_4 W'(x) = 0, \tag{3.3}$$

$$-\alpha_2\lambda^2 W(x) + \alpha_4\Theta'(x) - \alpha_4 W''(x) = F_1. \tag{3.4}$$

We first solve the associated homogeneous equations, in which $F_1 = 0$.

We set the trial solution as follows,

$$W_h(x) = B_w \exp(-px),\ \Theta_h(x) = B_\theta \exp(-px), \tag{3.5}$$

where B_w and B_θ are amplitude and p may be real, complex, or purely imaginary with any sign.

Substituting Eq. (3.5) into Eqs. (3.3), (3.4) gives

$$\begin{bmatrix} -(\alpha_3 + i\lambda\eta)p^2 - \alpha_1\lambda^2 + \alpha_4 & \alpha_4 p \\ -\alpha_4 p & -\alpha_4 p^2 - \alpha_2\lambda^2 \end{bmatrix} \begin{bmatrix} B_\theta \\ B_w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.6}$$

Eq. (3.6) may be written in matrix form as

$$\mathbf{CB} = 0. \tag{3.7}$$

The necessary and sufficient condition for non-zero elements in the column vector B of Eq. (3.7) is that the determinant of the coefficient matrix **C** shall be zero. Hence, the characteristic equation which correspond to the solution for the non-trivial case is given by

$$(-(\alpha_3 + i\lambda\eta)p^2 - \alpha_1\lambda^2 + \alpha_4)(-\alpha_4 p^2 - \alpha_2\lambda^2) + (\alpha_4 p)^2 = 0 \tag{3.8}$$

or,

$$(\alpha_3\alpha_5 - i\eta\lambda\alpha_4)p^4 + (\alpha_1\alpha_4\lambda^2 + \alpha_2\alpha_3\lambda^2 - i\eta\lambda^3\alpha_2)p^2 + (-\alpha_2\alpha_4\lambda^2 + \alpha_1\alpha_2\lambda^4) = 0\,. \tag{3.9}$$

The characteristic Eq. (3.9) shows that for any λ, there are two corresponding values of p^2, one positive and the other negative. The symbolic computing of MATHEMATICA is used to solve the above equation.

We write these values as

$$p_{11} = e_1 + ie_2, \quad p_{12} = -(e_1 + ie_2), \quad p_{21} = e_3 + ie_4, \quad p_{22} = -(e_3 + ie_4)\,. \tag{3.10}$$

where the constants e_i are positive real quantities presented in Appendix A. The symbol e_2 and e_3 represent real numbers attributable to damping. If damping is zero, these values are expressed as

$$p_{11} = e_1, \quad p_{12} = -e_1, \quad p_{21} = ie_4, \quad p_{22} = -ie_4\,. \tag{3.11}$$

Substituting Eq. (3.10) into Eq. (3.6), we obtain the corresponding amplitude ratios

$$r_{jk} = \frac{\alpha_4 p_{jk}}{(\alpha_3 + i\lambda\eta)p_{jk}{}^2 + \alpha_1\lambda^2 - \alpha_4}, \tag{3.12}$$

where j =1 or 2 and k =1 or 2.

Then the homogeneous solutions may be written as

$$w_h(x,t) = W_h(x)\exp(i\lambda t),\ \theta_h(x,t) = \Theta_h(x)\exp(i\lambda t), \tag{3.13}$$

where the mode functions are

$$W_h(x) = A_{11}e^{p_{11}x} + A_{12}e^{p_{12}x} + A_{21}e^{p_{21}x} + A_{22}e^{p_{22}x}, \tag{3.14}$$

$$\Theta_h(x) = r_{11}A_{11}e^{p_{11}x} + r_{12}A_{12}e^{p_{12}x} + r_{21}A_{21}e^{p_{21}x} + r_{22}A_{22}e^{p_{22}x}. \tag{3.15}$$

The coefficients A_{jk} are complex numbers that must be determined from boundary conditions.

The particular solutions of Eq. (3.4) is expressed by

$$W_p(x) = -\frac{F_1}{\alpha_2\lambda^2}, \tag{3.16}$$

$$\Theta_p(x) = 0. \tag{3.17}$$

Hence, the general solution of the Eqs. (2.46) and (2.47) are

$$w = (W_h(x) + W_p(x) = A_{11}e^{p_{11}x} + A_{12}e^{p_{12}x} + A_{21}e^{p_{21}x} + A_{22}e^{p_{22}x} - \frac{F_1}{\alpha_2\lambda^2})e^{i\lambda t}, \tag{3.18}$$

$$\theta = (r_{11}A_{11}e^{p_{11}x} + r_{12}A_{12}e^{p_{12}x} + r_{21}A_{21}e^{p_{21}x} + r_{22}A_{22}e^{p_{22}x})e^{i\lambda t}. \tag{3.19}$$

The displacement w and angular displacement θ must be defined as a function of the input frequency λ. There are four coefficients A_{jk} to determine. It still remains to satisfy the boundary conditions, which we take to be at $x=0$ and $x=L$. There are a total of four homogeneous boundary conditions, two from each end according to the alternatives in Eqs. (2.54) and (2.55). The conditions at $x=0$ require that

$$A_{11}+A_{12}+A_{21}+A_{22}-\frac{F_1}{\alpha_2\lambda^2}=0\,, \tag{3.20}$$

$$A_{11}p_{11}+A_{12}p_{12}+A_{21}p_{21}+A_{22}p_{22}=0\,. \tag{3.21}$$

The boundary conditions at $x=L$ require

$$r_{11}A_{11}p_{11}e^{p_{11}L}+r_{12}A_{12}p_{12}e^{p_{12}L}+r_{21}A_{21}p_{21}e^{p_{21}L}+r_{22}A_{22}p_{22}e^{p_{22}L}=0\,, \tag{3.22}$$

$$(r_{11}-p_{11})A_{11}e^{p_{11}L}+(r_{12}-p_{12)}A_{12}e^{p_{12}L}+(r_{21}-p_{21})A_{21}e^{p_{21}L}+(r_{22}-p_{22})A_{22}e^{p_{22}L}=0\,. \tag{3.23}$$

Four coefficients A_{jk} can be solved from Eqs. (3.20)- (3.23) by using MATHMATICA.

3.2 Case II: Concentrated load at the free end

Let the beam in Fig. (4) be subjected to a general harmonic excitation carrying a concentrated load of intensity F_2 at the free end. So we defined new boundary conditions including shear force at the end of the beam.

$$\theta'\,|_{x=L}=0\,,\ \alpha_4(w'-\theta)\,|_{x=L}=F_2\,. \tag{3.24}$$

We put $F_1=0$ and define the corresponding equations substituting Eqs. (3.20) and (3.23) to solve the unknown constants as follows

$$A_{11}+A_{12}+A_{21}+A_{22}=0\,, \tag{3.25}$$

$$(r_{11}-p_{11})A_{11}e^{p_{11}L}+(r_{12}-p_{12})A_{12}e^{p_{12}L}+(r_{21}-p_{21})A_{21}e^{p_{21}L}+(r_{22}-p_{22})A_{22}e^{p_{22}L}+\frac{F_2}{\alpha_4}=0\,, \tag{3.26}$$

The symbolic computation is used to solve the above system of equations.

CHAPTER IV

RESULTS OF CALCULATION

In order to validate and confirm the effect of energy dissipation, the exact expressions for the frequency equations and mode shapes given by Eqs. (3.18) and (3.19) programmed to compute the amplitude of a cantilever beam. The geometric and material parameters used in these examples are: h_1 =0.0005 m, h_2 =0.004 m, ρ_1 =7.8 $\times 10^3\, kg/m^3$, ρ_2 =48 kg/m^3 , C_{66} =48MPa, λ =48.48kPa, μ =27.27 kPa, $\nu = 0.28$, L=1 m; these properties of core and faces material are taken from ECA honeycomb and steel, respectively.

Table 5.1 Constants of sixth order differential equation in W(x)

η	Present Theory			Nilsson		
	A_1	A_2	A_3	$\bar{A}_1$	$\bar{A}_2$	$\bar{A}_1$
0	-6401	-1523	39959	-1593	-244	16165
0.01	-6401+0.1i	-1523-10i	39959+15i	-1610+16i	-244+2i	16322-33i
0.05	-6401+0.7i	-1523-50i	39959+76i	-1669+83i	-243.9+12i	16847-169i

From Table 5.1, we can find the real numbers have same sign and imaginary number signs are different in each case. In Figs. 5-12, we can observe the changes of local maximum amplitude in beam vibration with uniform load along the horizontal. It is evident the attenuation rate of amplitude at each mode increases drastically as damping is increased, but damped period is affected much less. Note that the amplitude in higher modes diminished first as damping is increased. The reason is that we assumed the rate of damping in Eq. (2.33) as a function of axial displacement. Physically in beam vibration, the dominant degree of freedom in lower modes is a transverse displacement, *w*. According to our defined dissipation model, the amplitude of transverse displacement mainly diminished in higher modes. In Fig. 10, under fixed frequency, the amplitude of first mode is also diminished as damping is increased because the variables of θ and w are coupled in governing equations of the system.

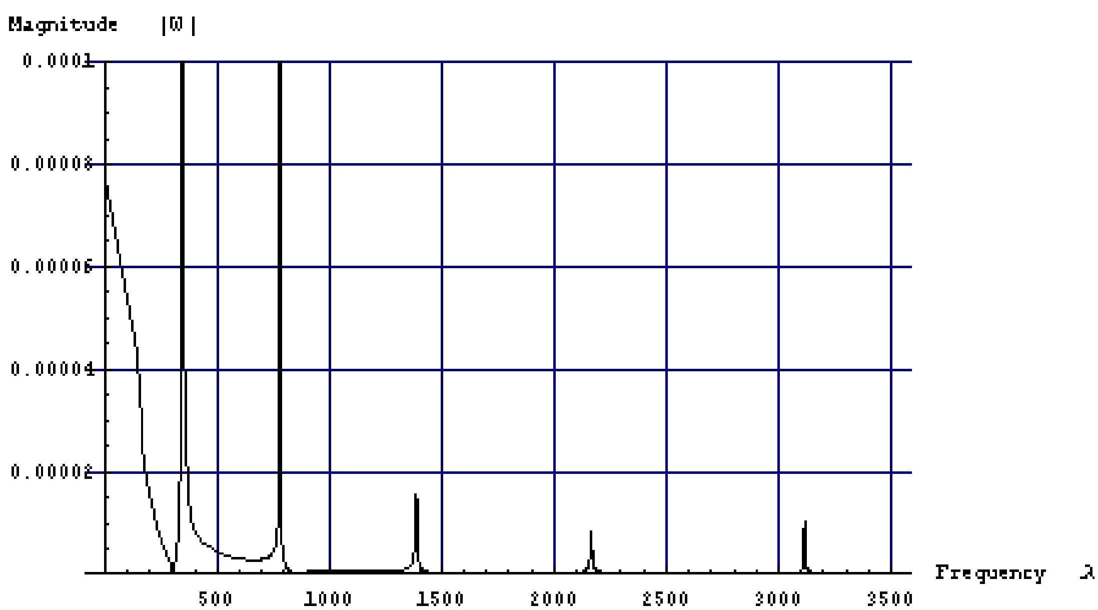

Fig. 5. Amplitude of W versus frequency (Case I: $F_1 = 10, \eta = 0$)

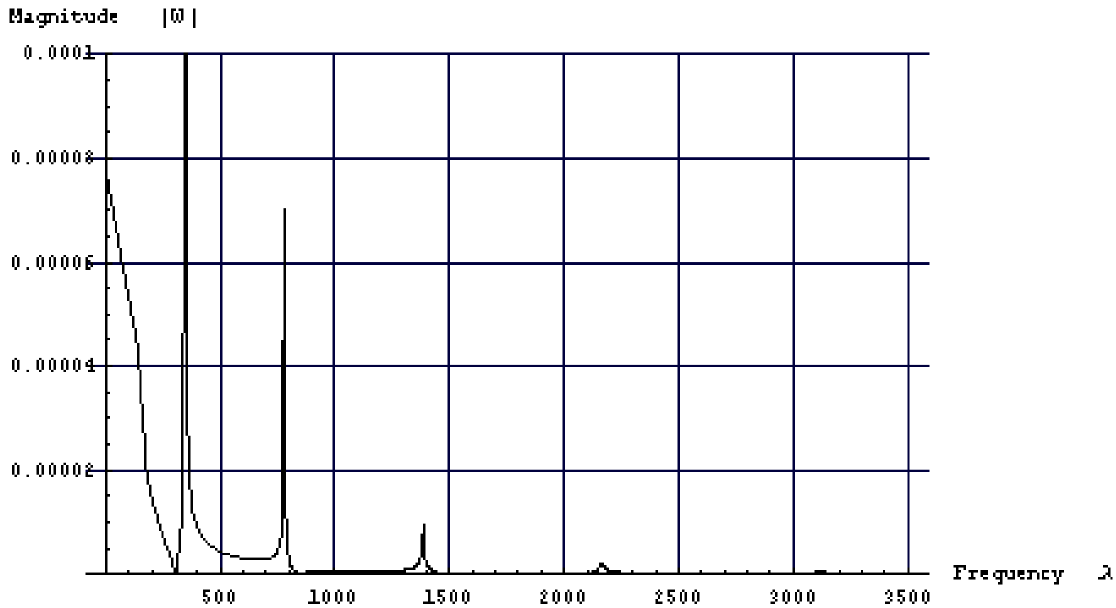

Fig. 6. Amplitude of W versus frequency (Case I: $F_1 = 10, \eta = 0.01$)

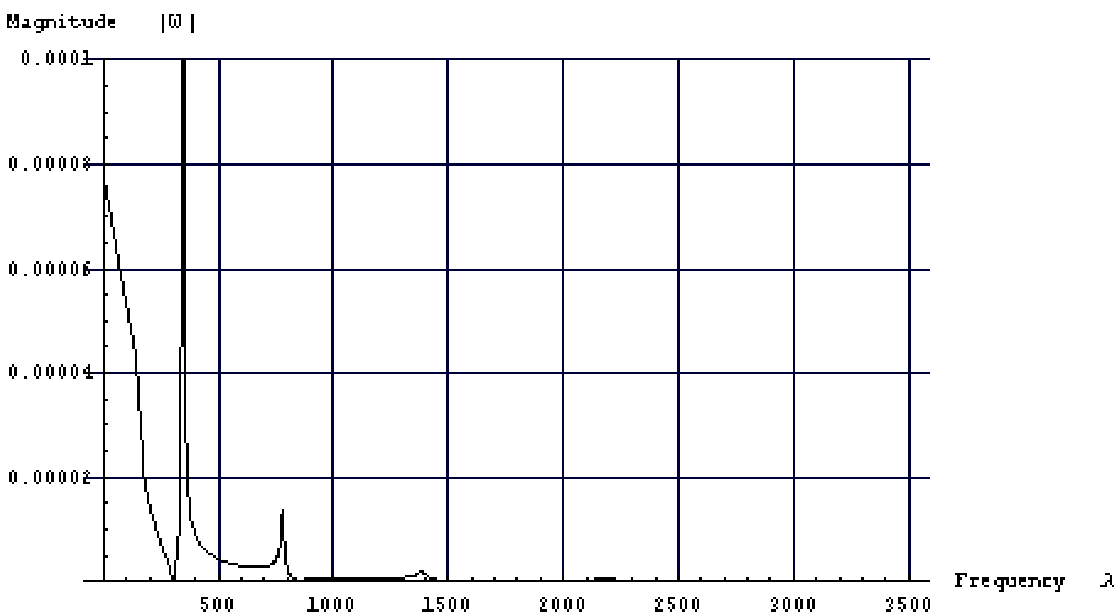

Fig. 7. Amplitude of W versus frequency (Case I: $F_1 = 10, \eta = 0.05$)

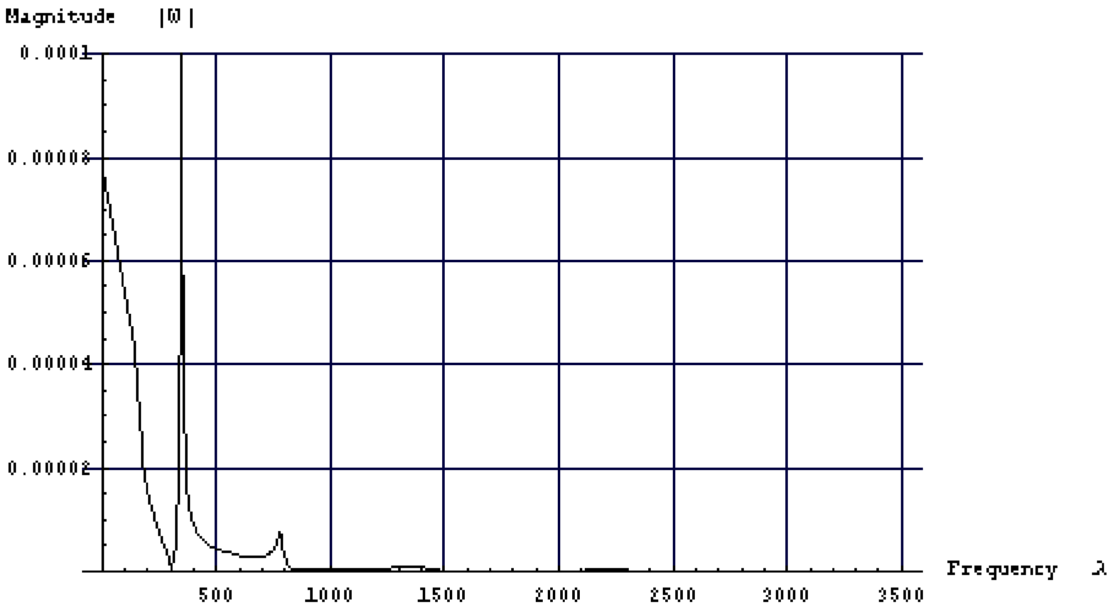

Fig. 8. Amplitude of *W* versus frequency (Case I: $F_1 = 10, \eta = 0.1$)

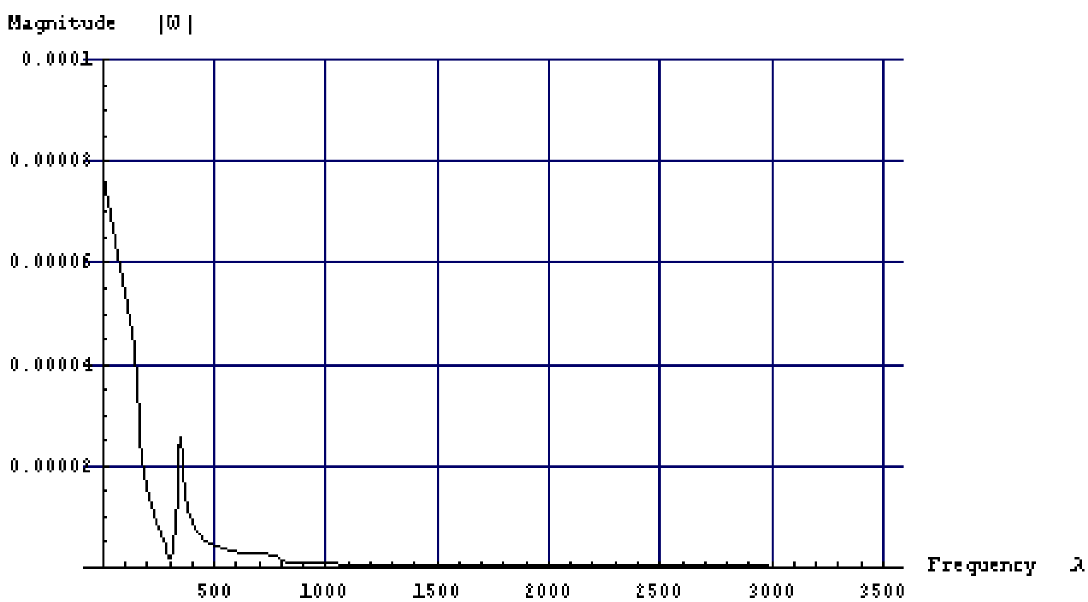

Fig. 9. Amplitude of *W* versus frequency (Case I: $F_1 = 10, \eta = 0.5$)

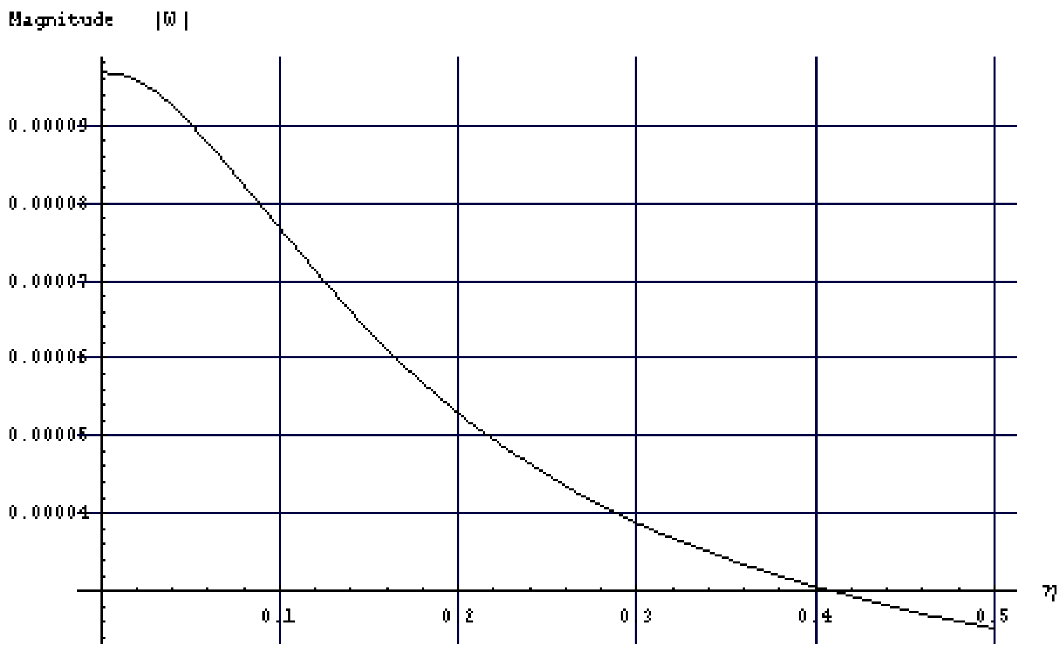

Fig. 10. Amplitude of W versus damping constant with $\lambda = 350$ (Case I: $F_1 = 10$)

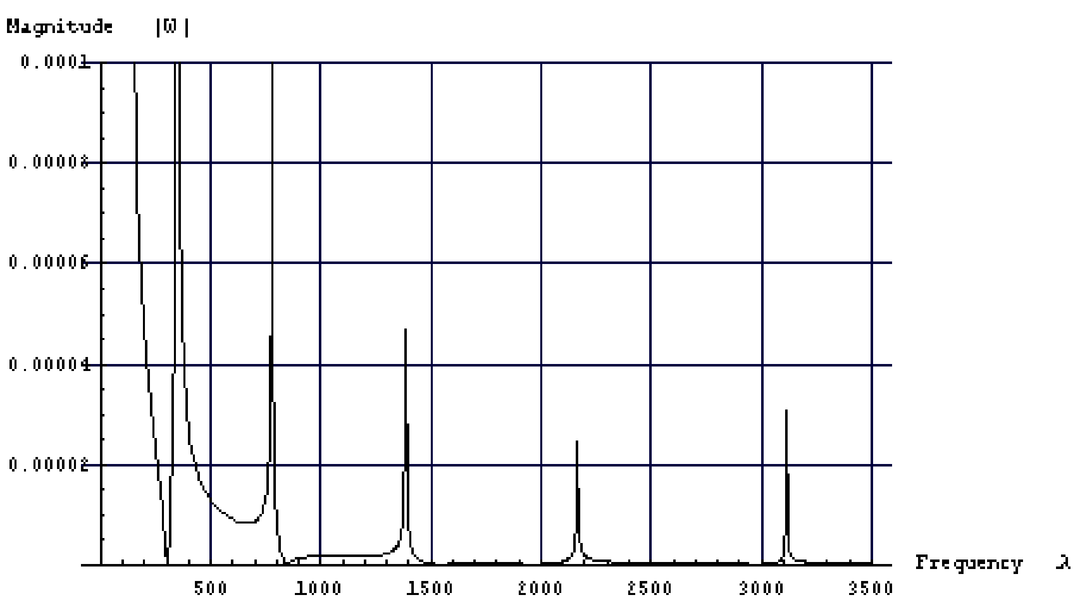

Fig. 11. Amplitude of W versus frequency (Case I: $F_1 = 30, \eta = 0$)

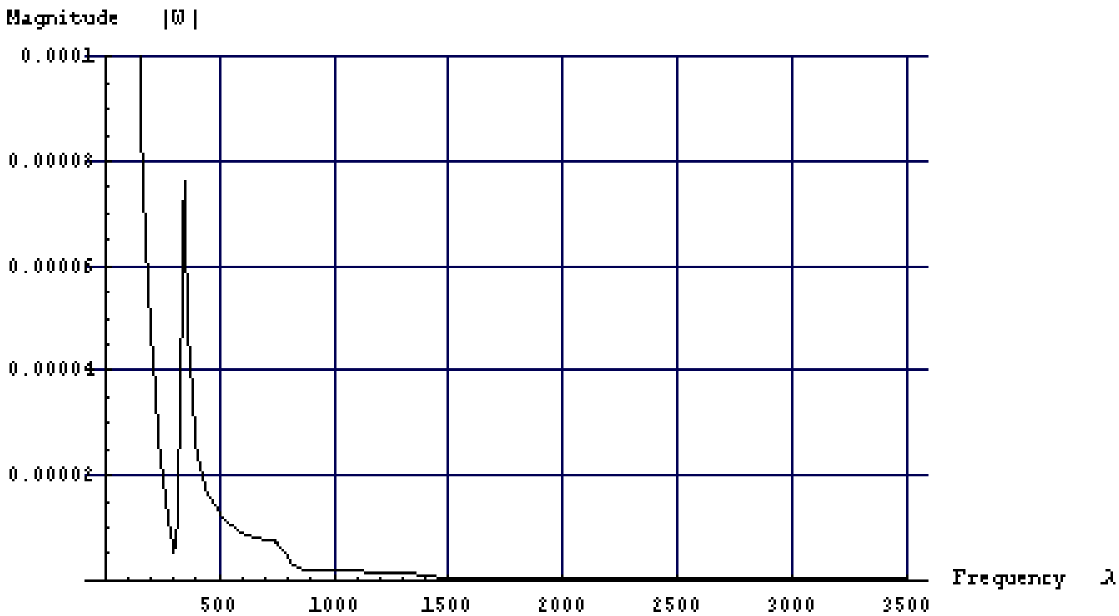

Fig. 12. Amplitude of W versus frequency (Case I: $F_1 = 30, \eta = 0.5$)

Figs. 11 and 12 show that the amplitude is directly proportional to the applied force, and the former increases with the latter. But the applied force does not influence the natural frequency modes and values.

In Figs. 13-18, we can observe the changes of local maximum amplitude in beam vibration with a concentrated load of intensity F_2 at the free end. There is similarity between two cases that the amplitude in higher modes diminished first as damping is increased. But the natural frequency is changed due to different boundary condition in Eqs. (3.25) and (3.26).

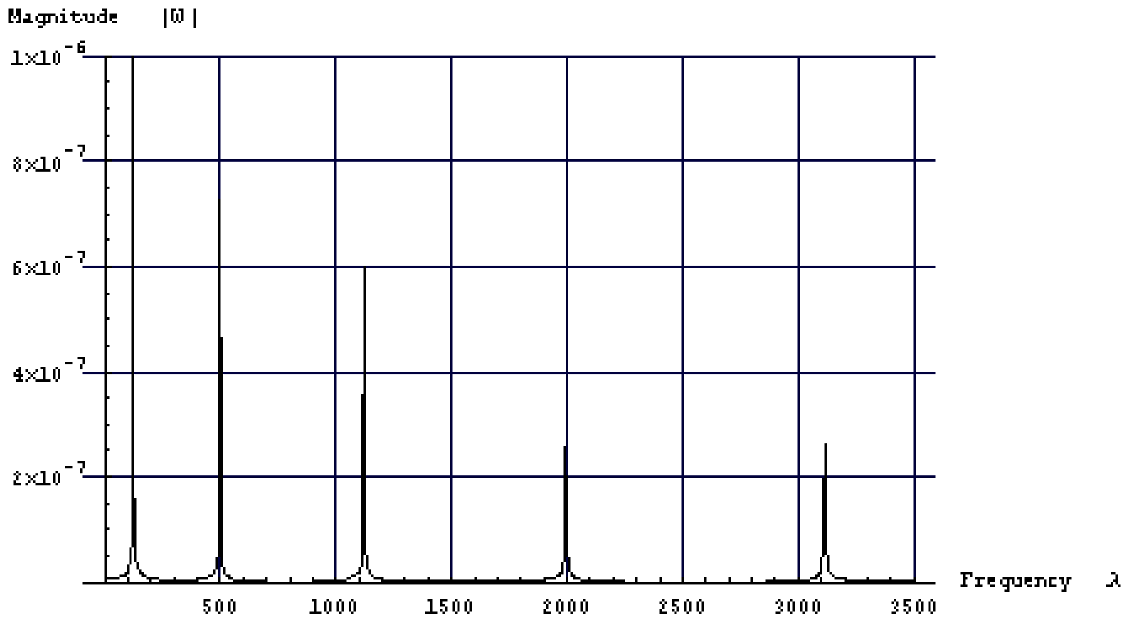

Fig. 13. Amplitude of W versus frequency (Case II: $F_2 = 10, \eta = 0$)

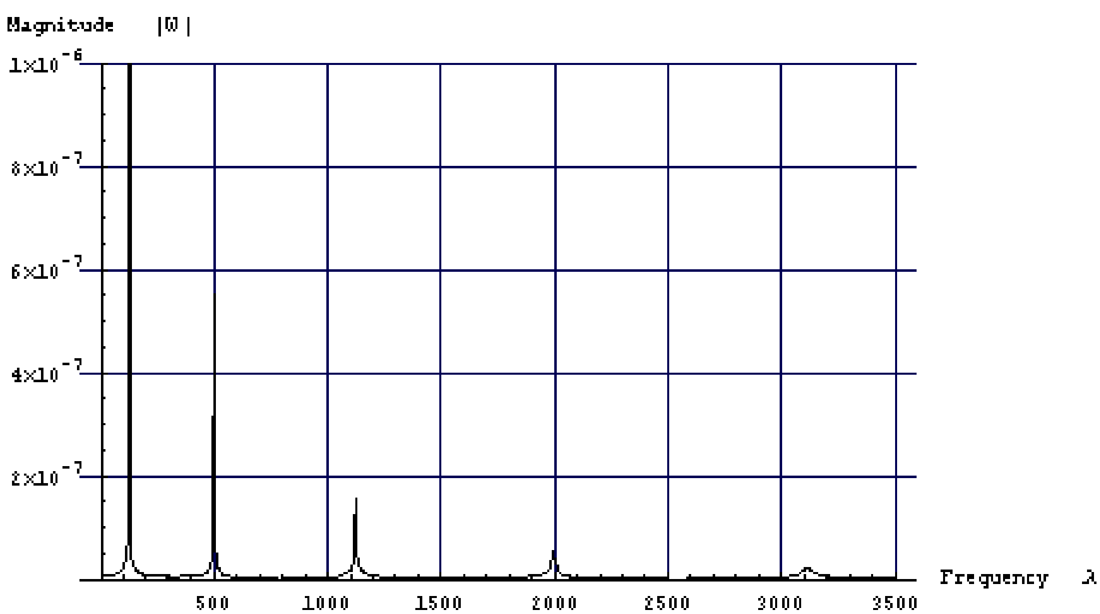

Fig. 14. Amplitude of W versus frequency (Case II: $F_2 = 10, \eta = 0.01$)

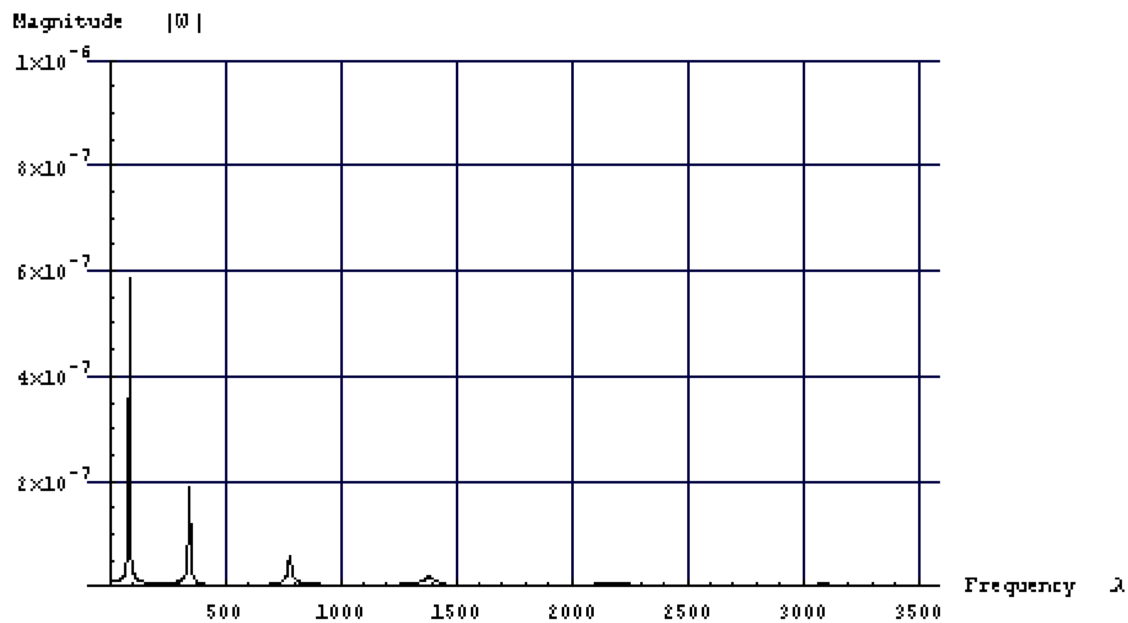

Fig. 15. Amplitude of W versus frequency (Case II: $F_2 = 10, \eta = 0.05$)

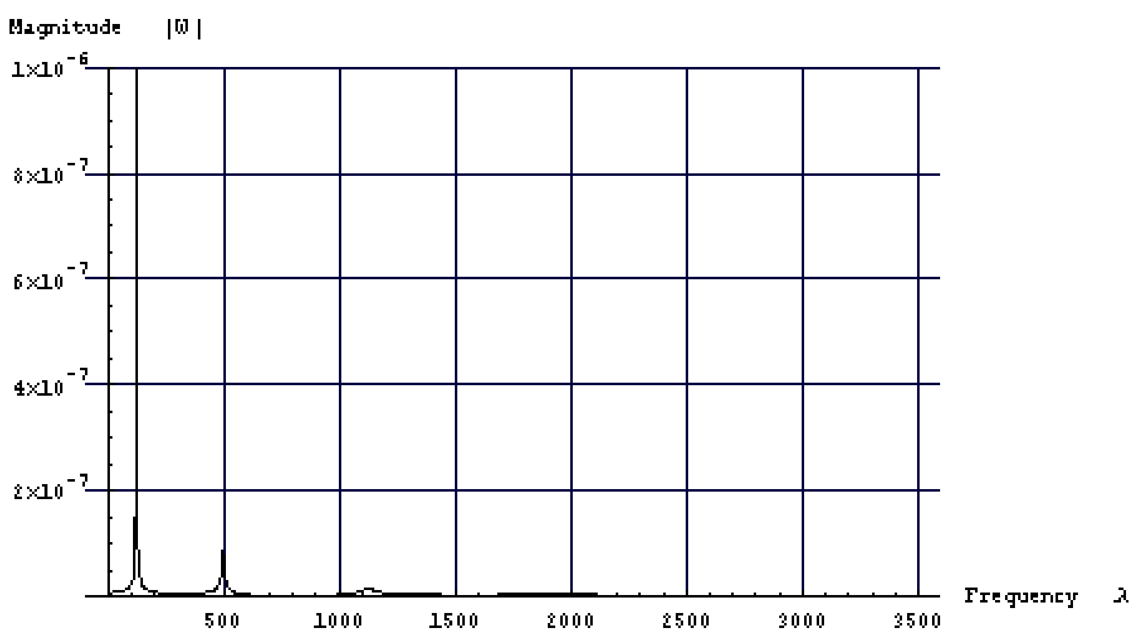

Fig. 16. Amplitude of W versus frequency (Case II: $F_2 = 10, \eta = 0.1$)

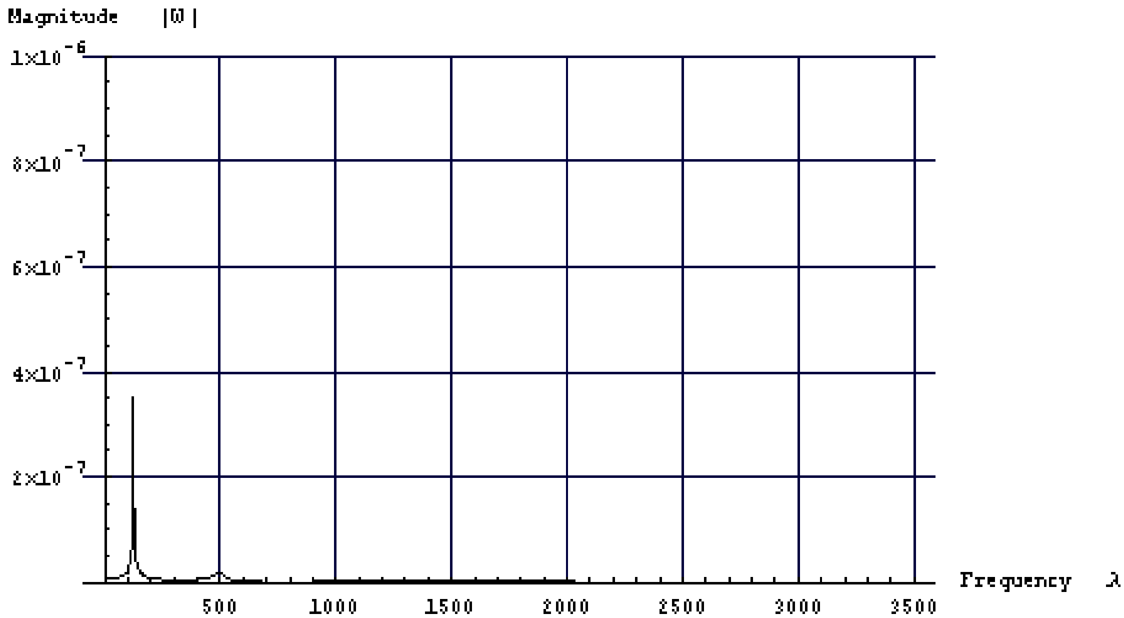

Fig. 17. Amplitude of *W* versus frequency (Case II: $F_2 = 10, \eta = 0.5$)

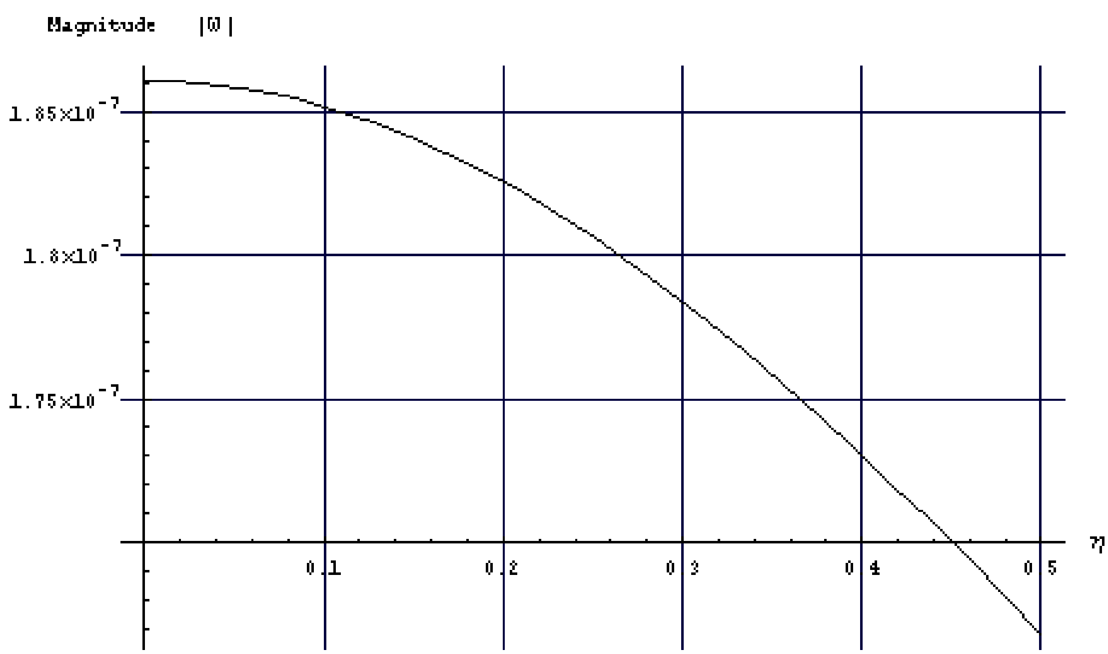

Fig. 18. Amplitude of *W* versus damping constant with $\lambda = 122$ (Case II: $F_2 = 10$)

CHAPTER V

CONCLUSION

Analytical expressions for the frequency equation and mode shapes of a sandwich beam that takes into consideration influence of energy dissipation is presented in explicit form. The assumption of maximum rate of dissipation is applied to derive the force-displacement relations. Then the system of differential equations for a given sandwich beam is obtained using equilibrium equations.

The influence of the harmonic excitation carrying a uniform load along the horizontal and concentrated load at the free end was also determined. From the case study it can be seen that proposed analytical method is reliable method that offers researchers an approach in the analysis of energy dissipation in sandwich beam. It can also be seen that results from analytical method can be successfully used to illustrate and explain the behaviors of damping effect to the amplitude in sandwich beam.

In the frequency response analysis, the amplitude is found to be directly proportional to the applied force, but the applied force does not influence the natural frequency modes and values. The influence of amplitude due to assumed dissipation function from angular displacement is dominant in higher mode.

In future work, it will be worthwhile to consider smart composite beam structure with embedded piezoelectric actuator or ER fluid as dissipation materials.

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APPENDIX

CONSTANTS

The constants e_i are given by

$$e_1 = \sqrt{\frac{d_1 + \sqrt{d_1^2 + d_2^2}}{2}}, \quad e_2 = \frac{\sqrt{2}\, d_2}{2\sqrt{d_1 + \sqrt{d_1^2 + d_2^2}}},$$

$$e_3 = \frac{\sqrt{2}\, d_4}{2\sqrt{d_3 + \sqrt{d_3^2 + d_4^2}}}, \quad e_4 = \sqrt{\frac{d_3 + \sqrt{d_3^2 + d_4^2}}{2}},$$

where

$$d_1 = \frac{-\lambda^2 f_1 f_7 + b_1 f_7 - \lambda^2 f_2 f_8 + b_2 f_8}{f_7^2 + f_8^2},$$

$$d_2 = \frac{-\lambda^2 f_1 f_7 - b_1 f_7 - \lambda^2 f_2 f_8 - b_2 f_8}{f_7^2 + f_8^2},$$

$$d_3 = \frac{\lambda^2 f_1 f_8 - b_1 f_8 - \lambda^2 f_2 f_7 + b_2 f_7}{f_7^2 + f_8^2},$$

$$d_4 = \frac{\lambda^2 f_1 f_8 + b_1 f_8 - \lambda^2 f_2 f_7 - b_2 f_7}{f_7^2 + f_8^2},$$

where

$$b_1 = \sqrt{\frac{a_1 + \sqrt{a_1^2 + a_2^2}}{2}}, \quad b_2 = \frac{\sqrt{2}\, a_2}{2\sqrt{a_1 + \sqrt{a_1^2 + a_2^2}}},$$

$$a_1 = \lambda^4 (f_1^2 - f_2^2) + f_3 f_4 f_6, \quad a_2 = 2\lambda^4 f_1 f_2 - f_3 f_4 f_6,$$

with

$$f_1 = \alpha_1\alpha_4 + \alpha_2\alpha_3,$$

$$f_2 = -\alpha_2\eta\lambda,$$

$$f_3 = 4\alpha_2\alpha_3\lambda^2,$$

$$f_4 = \alpha_3,$$

$$f_5 = \frac{1}{2}\eta\lambda,$$

$$f_6 = \alpha_4 - \frac{1}{2}\alpha_1\lambda^2$$

$$f_7 = 4\alpha_3\alpha_4,$$

$$f_8 = \alpha_1\lambda^2$$

and α_1, α_2, α_3, and α_4 are given by Eqs. (2.15), (2.16), (3.21) and (3.22).

VITA

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